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LETTER TO THE EDITOR

A dynamical model for the origin of Snyder's quantized spacetime algebra

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Abstract. We discuss a dynamical model in which Dirac's constraint theory generates a continuum of possible dynamics algebras. Upon canonical quantization almost all are found to violate the Jacobi identity. Of the two anomaly-free algebras, one is Snyder's quantized spacetime algebra and the other is a curved momentum space algebra.

In 1947 Snyder [1, 2] wrote down a realization of an operator algebra which he interpreted as a model for quantized relativistic spacetime coordinates x^μ, $\mu = 0, 1, 2, 3$. His method was to use real coordinates η_0, η_1, η_2, η_3, η_4 in a five-dimensional space V^5 with a Lorentz signature metric projecting onto a four-dimensional spacetime of constant curvature, that is, a De Sitter space associated with a fundamental length a. In his model, the standard (i.e. commuting) spacetime coordinate algebra is recovered in the limit $a \to 0$. The operators x^μ are defined without further motivation by

$$x^\mu = \mathrm{i}a(\eta^\mu \partial^4 - \eta^4 \partial^\mu) \qquad \mu = 0, 1, 2, 3 \tag{1}$$

where components with upper indices in V^5 are obtained using the metric tensor as an index raising operator. In the following we shall take $c = \hbar = 1$.

Snyder also introduced the variables $p_\mu = (1/a)\eta_\mu/\eta_4$, interpreted by him as operators of energy and momentum, so that the full operator algebra takes the form

$$\begin{aligned} &[x^\mu, x^\nu] = \mathrm{i}a^2 M^{\mu\nu} \\ &[p^\mu, x^\nu] = \mathrm{i}(\eta^{\mu\nu} - \mathrm{i}a^2 p^\mu p^\nu) \\ &[p^\mu, p^\nu] = 0 \end{aligned} \tag{2}$$

where $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and $\eta^{\mu\nu}$ are the components of the metric tensor, with $\eta^{00} = 1$, $\eta^{11} = \eta^{22} = \eta^{33} = -1$. The $M^{\mu\nu}$ satisfy the same algebra as the generators of angular momentum and boosts in the standard theory.

An unsatisfactory feature of Snyder's paper is the *ad hoc* way in which the operator algebra is simply postulated; it would be preferable to arrive at such an algebra in a more natural way. In this letter we discuss a point particle model in which the same dynamical algebra occurs as a natural consequence of Dirac's constraint theory applied to a system with second-class constraints. An advantage is that we can dispense with the five-dimensional space coordinates η^A, $A = 0, 1, 2, 3, 4$.

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The dynamical degrees of freedom in our model are four spacetime coordinates x^μ, $\mu = 0, 1, 2, 3$ plus one extra coordinate z. We will also use variables e and λ at the start to generate constraints, but these variables together with z drop out in the final analysis. All coordinates are taken to be real and bosonic. The Lagrangian for the system is given by

$$L = \frac{e}{2}(\dot{x}^\mu \dot{x}_\mu + \epsilon \dot{z}^2) + \frac{\alpha}{2e} + \lambda(x^\mu x_\mu + \epsilon z^2 - \beta) \tag{3}$$

where α and β are constants and $\epsilon = \pm 1$. The signs of these constants are left unspecified at this stage.

By inspection, it will be seen that the role of the variable e is to generate the analogue of the mass–shell constraint in conventional relativistic point particle models, whereas the role of the variable λ is to constrain the coordinates x^μ and z to a four-dimensional subspace of constant curvature. Following the standard constraint analysis of Dirac [3] we find the three constraints

$$\begin{aligned} \phi_1 &\equiv p^\mu p_\mu + \epsilon \pi^2 - \alpha \approx 0 \\ \phi_2 &\equiv x^\mu x_\mu + \epsilon z^2 - \beta \approx 0 \\ \phi_2 &\equiv x^\mu p_\mu + z\pi \approx 0 \end{aligned} \tag{4}$$

where p_μ and π are the momenta conjugate to x^μ and z respectively. The fundamental Poisson brackets are

$$\begin{aligned} \{p_\mu, x^\nu\}_{\mathrm{PB}} &= \delta_\mu^\nu \\ \{\pi, z\}_{\mathrm{PB}} &= 1. \end{aligned} \tag{5}$$

These constraints are stable under temporal evolution of the system and have the following Poisson brackets with each other on the surface of constraints:

$$\begin{aligned} \{\phi_1, \phi_2\}_{\mathrm{PB}} &\approx 0 \\ \{\phi_1, \phi_3\}_{\mathrm{PB}} &\approx 2\alpha \\ \{\phi_2, \phi_3\}_{\mathrm{PB}} &\approx -2\beta. \end{aligned} \tag{6}$$

By inspection we deduce that there are two second-class constraints χ_1, χ_2 and one first-class constraint Φ in the system. The latter is a linear combination of the above constraints such that it has a zero Poisson bracket with all the other constraints, i.e. we look for constants κ_1, κ_2, κ_3 such that $\Phi = \kappa_1\phi_1 + \kappa_2\phi_2 + \kappa_3\phi_3$ satisfies $\{\Phi, \phi_i\}_{\mathrm{PB}} = 0$, $i = 1, 2, 3$. A solution is $\Phi = \beta\phi_1 + \alpha\phi_2$, which turns out to be proportional to the total Hamiltonian. We note that this Hamiltonian vanishes on the surface of constraints, as expected in a generally covariant theory.

For the pair of second-class constraints we define

$$\chi_1 := f\phi_1 - g\phi_2 \qquad \chi_2 := \phi_3 \tag{7}$$

where f and g are any functions on a phase space which satisfy the condition that on the surface of constraints, $h := \alpha f + \beta g$ does not vanish. We shall take f and g to be constants. It turns out that Snyder's algebra coincides with the choice $g = 0$. Provided $h \neq 0$ then

χ_1 and χ_2 are indeed second class. Using these we may construct Dirac brackets [3] in the standard way to find

$$\begin{aligned}
\{x^\mu, x^\nu\}_{\mathrm{DB}} &= \frac{f}{h}(x^\mu p^\nu - x^\nu p^\mu) \\
\{p^\mu, x^\nu\}_{\mathrm{DB}} &= \eta^{\mu\nu} - \frac{1}{h}(g x^\mu x^\nu - f p^\mu p^\nu) \\
\{p^\mu, p^\nu\}_{\mathrm{DB}} &= \frac{g}{h}(x^\mu p^\nu - x^\nu p^\mu).
\end{aligned} \tag{8}$$

When the second-class constraints are used with the Dirac brackets replacing the Poisson brackets we recover equations of motion equivalent to those obtained via the Euler–Lagrange equations of motion. We note that the Dirac brackets between z, π, x^μ and p^μ are non-zero.

Canonical quantization follows the standard prescription

$$[\hat{A}, \hat{B}] = \mathrm{i}\{A, B\}_{\mathrm{DB}} \tag{9}$$

leading to the algebra

$$\begin{aligned}
[\hat{x}^\mu, \hat{x}^\nu] &= \frac{\mathrm{i}f}{2h}[\hat{x}^\mu \hat{p}^\nu + \hat{p}^\nu \hat{x}^\mu - \hat{x}^\nu \hat{p}^\mu - \hat{p}^\mu \hat{x}^\nu] \\
[\hat{p}^\mu, \hat{x}^\nu] &= \mathrm{i}\eta^{\mu\nu} - \frac{g}{2h}[\hat{x}^\mu \hat{x}^\nu + \hat{x}^\nu \hat{x}^\mu] - \frac{f}{2h}[\hat{p}^\mu \hat{p}^\nu + \hat{p}^\nu \hat{p}^\mu] \\
[\hat{p}^\mu, \hat{p}^\nu] &= \frac{\mathrm{i}g}{2h}[\hat{x}^\mu \hat{p}^\nu + \hat{p}^\nu \hat{x}^\mu - \hat{x}^\nu \hat{p}^\mu - \hat{p}^\mu \hat{x}^\nu]
\end{aligned} \tag{10}$$

where we have symmetrized operator products in the conventional way.

We now turn to the question of the choice of algebra for physical applications. A fundamental requirement for consistency is that the Jacobi identity should be satisfied. First consider the classical Dirac bracket algebra (8). This has been evaluated on the surface of constraints. If we now test for the Jacobi identity

$$\{x^\mu, \{x^\nu, x^\lambda\}_{\mathrm{DB}}\}_{\mathrm{DB}} + \{x^\nu, \{x^\lambda, x^\mu\}_{\mathrm{DB}}\}_{\mathrm{DB}} + \{x^\lambda, \{x^\mu, x^\nu\}_{\mathrm{DB}}\}_{\mathrm{DB}} = 0 \tag{11}$$

we find that this equation is not satisfied for some values of the indices. Similar results hold using the momentum coordinates. In each case the left-hand side is proportional to fg.

We may refer to this failure as the classical Jacobi anomaly. The reason for this is readily understood when we recall the essential point emphasized by Dirac [3]: the process of taking Poisson brackets does not commute with the process of applying constraints. The classical Jacobi anomaly does not occur only when we apply the second-class constraints χ_1, χ_2 *after* evaluating all partial derivatives in the Jacobi identity calculation for the Dirac brackets. This result has been verified for all possible combinations of the coordinates and momenta using the MAPLE symbolic algebra manipulation package.

Because the quantized algebra (10) was extracted from (8), which itself does not satisfy the classical Jacobi identity, it should not surprise us if the quantum operator Jacobi identity does not hold for (10) either. This indeed turns out to be the case. The proposed commutators (10) generate bilinear combinations of the operators which do not readily simplify in the Jacobi identity calculation. Using MAPLE we were able to find cancellations for most but not all combinations of indices. This is not a direct proof that the quantum

Jacobi identity does not hold, but the pattern of the calculations makes this most probably the case. As with the classical Jacobi anomaly, the quantum Jacobi anomaly is proportional to fg at the first level of calculation. Successive substitutions of operator products generate terms proportional to higher powers of fg. This leads us to believe that the quantum anomaly does exist if $fg \neq 0$. It is certainly the case that the Jacobi anomaly in either its classical or quantum form does not occur if either $f = 0$ or $g = 0$.

We argued above that the classical Jacobi anomaly can be understood by the failure of the processes of differentiation and taking of constraints to commute. We may quite naturally ask for a similar explanation of the quantum Jacobi anomaly. A serious problem arises here, however, because the Jacobi identity for operators really is an identity, involving a direct cancellation of operator products regardless of any particular representation of the operators. We, therefore, appear to have a hard task trying to explain the quantum Jacobi anomaly. The only answer which comes to mind is that for $fg \neq 0$ there does not exist any representation of the 'algebra' in terms of associative linear operators; the inexorable logic of the quantum Jacobi identity would otherwise hold and we could immediately prove that there was no quantum anomaly.

For $g = 0$ we recover Snyder's spacetime–momentum algebra if we identify the constant α with Snyder's constant a^{-2} and change the sign of the energy p_t [1]. Physically, we would expect Snyder's a to be related to the cosmological constant, that is, virtually zero, and then the spacetime algebra becomes consistent with the normal Poincaré algebra. The mass–shell condition for a finite rest mass will still hold provided that the quantity $\pi^2 - \alpha$ remains finite as α tends to infinity.

Using MAPLE, we were able to prove that the abstract Snyder's algebra does indeed satisfy the quantum Jacobi identity and we must, therefore, be able to find at least one representation for the operators. Snyder gave a representation which in our notion reduces to the form

$$\begin{aligned} &\hat{p}^\mu \to p^\mu \\ &\hat{x}^\mu \to -\mathrm{i}\frac{\partial}{\partial p_\mu} + \mathrm{i} f p^\mu p_\nu \frac{\partial}{\partial p_\nu}. \end{aligned} \tag{12}$$

Using MAPLE we were indeed able to prove that this particular representation satisfies the quantum Jacobi identity.

It is of interest that another solution to the problem of the anomaly is to take $f = 0$. Then we recover the algebra

$$\begin{aligned} &[x^\mu, x^\nu] = 0 \\ &[p^\mu, x^\nu] = \mathrm{i}\left(\eta^{\mu\nu} - \frac{\mathrm{i}}{\beta} x^\mu x^\nu\right) \\ &[p^\mu, p^\nu] = \frac{\mathrm{i}}{\beta} M^{\mu\nu}. \end{aligned} \tag{13}$$

This algebra is isomorphic to Snyder's algebra and so it satisfies the quantum Jacobi identity and has an analogous representation. In this case we interpret the algebra as corresponding to a curved momentum space. Work is in hand on the further interpretation and application of our results to the problem of regularization of field theory. The same curved momentum space was discussed recently by Mir-Kasimov using quantum-group theoretical methods [4], showing how a curved momentum space approach can be used to smooth out singularities in field theory propagators.

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References

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